COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 163

ON THE CONCEPT OF OPTIMAL ECONOMIC GROWTH

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December 6, 1963

ON THE CONCEPT OF OPTIMAL ECONOMIC GROWTH*

by

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1. Approaches in the literature

The search for a principle from which an "optimal" rate of economic growth can be deduced holds great fascination to economists. A variety of attitudes or approaches to this problem can be discerned in the literature.

One school of thought, represented among others by Professor Bauer [1957], favors that balance between the welfare of present and future generations that is implied in the spontaneous and individual savings decisions of the present generation. A policy implementing this preference would merely seek to arrange for tax collection and other government actions affecting the economy in such a way as to distort or amend the individual savings preferences as little as possible.

Contrasting with this view is the position, expressed among others quite explicitly by Professor Allais [1947, Ch. VI], that the balancing of

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* This paper was prepared for a Study Week organized by The Pontifical Academy of Sciences, October 7 - 14, 1963, on "The role of econometric analysis in economic development plans and in controlling economic fluctuations." It is also to be presented at a joint session of the American Economic Association and the Econometric Society on "Intertemporal Economic Theory" in the Boston Meetings, December 1963.

** I am indebted to E. S. Phelps and H. E. Scarf for valuable comments.

This research was undertaken by the Cowles Foundation for Research in Economics, under Task NR 047-006 with the Office of Naval Research, and completed under a grant from the National Science Foundation.

the interests of different generations is an ethical or political problem, in which the competitive market solution has no valid claim to moral superiority over other solutions that depend for their realization on action by the state. A more specific optimality concept is implied in the strictures of Professor Harrod [1948, p. 40] and of Frank Ramsey [1928, p. 543] against any discounting of future utilities. These authors leave little doubt that they regard only equal weights for the welfare of present and future generations as ethically defensible.

The purpose of the present paper is to do some "logical experiments," in which various mathematical forms of the optimality criterion are confronted with a very simple model of technology and of population growth, to see what their maximization leads to. Our study is similar in purpose to Ramsey's classical paper, and to Tinbergen's recent exploration [1960] of the same problem. The underlying idea of this exploratory approach is that the problem of optimal growth is too complicated, or at least too unfamiliar, for one to feel comfortable in making an entirely a priori choice of an optimality criterion before one knows the implications of alternative choices. One may wish to choose between principles on the basis of the results of their application. In order to do so, one first needs to know what these results are. This is an economic question logically prior to the ethical or political choice of a criterion.

What is a suitable mathematical formalization of the idea of an optimality criterion? The most basic notion is that of a preference ordering of growth paths. Such an ordering states for each pair of alternative growth paths whether they are equally good, and if not, which is preferred. Indifference, preference and preference-or-indifference are usually required to be transitive.

An important class of preference orderings is that representable* by

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* Conditions of continuity under which a given preference ordering permits such a representation have been studied by Wold [1943] and by Debreu [1954].

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a continuous preference function (utility function, indicator, etc.). A particular function which has been frequently used has the form

$$U = \sum_{t=1}^{\infty} \kappa^{t-1} u(x_t)$$

for consumption paths $(x_1, x_2, \ldots)$ of infinite duration with discrete time $t = 1, 2, \ldots$. This form can be interpreted as a discounted sum of future one-period utilities $u(x_t)$ with a discount factor of κ per period. This form has been derived by the present author* from postulates

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* Koopmans [1960], especially Section 14.

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expressing, among other requirements,

(a) noncomplementarity of consumption in any three subperiods into which the future may be partitioned,

(b) stationarity in the sense that the ordering of any two paths is not altered if both consumption sequences are postponed by one time unit and identical consumptions are inserted in the gaps so created in each path.

The utility function so obtained is "cardinal" only in the limited sense that the simple form of a discounted sum is conserved only by <u>linear</u> transformations of the utility scale. If below we occasionally use the expression "utility difference," "marginal utility," these must be interpreted as elliptic phrases referring to a preference indicator of particularly simple form. There is no intent to express a belief that, even in the absence of risk or uncertainty, "utility" itself becomes a measurable quantity as soon as a zero point and a unit are specified.

There still remains a gap between the derivation of the above utility function from the postulates referred to and its use in the present study: For present purposes a continuous time concept is more appropriate.

2. Plan of the Present Paper

We shall freely borrow from Phelps [1961] and others mentioned below the assumptions of the main model considered in Section 4, from Ramsey [1928] a device for maximizing utility over an infinite horizon without discounting, together with methods for applying the device, from Srinivasan [1962] and from Uzawa [1963] information about the results of maximizing a discounted sum of future consumptic[illegible] and from Inagaki [1963] results about the generalization of that problem to the case of predictable technological progress. If this particular brew has not been served before, it is not put together here for any novelty of the combination. Rather, our eclectic model appears to have in it the minimum collection of elements needed to serve the two main aims of the present paper.

The first aim is to illustrate the usefulness of the tools and concepts of mathematical programming in relation to the problem of optimal economic growth.

The second aim is to argue against the complete separation of the ethical or political choice of an objective function from the investigation of the set of technologically feasible paths. Our main conclusion will be that such a separation is not workable. Ignoring realities in adopting "principles" may lead one to search for a nonexistent optimum, or to adopt an "optimum" that is open to unanticipated objections.

In connection with the first aim, Section 3 recalls a few facts from linear and convex programming in a finite number of variables, that bear on the problem of optimum growth. The reading of this section is believed to be helpful rather than essential for what follows. Indeed, in most of its formulations, the problem of optimal growth is a special problem in mathematical programming. The main new element arises from the open-endedness of the future. If one adopts a finite time horizon, the choice of the terminal capital stock is as much a part of the problem to be solved as the choice of the path. Terminal capital, after all, represents the collection of paths beyond the horizon that it makes possible. An infinite horizon is therefore perhaps a more natural specification in many formulations of the problem of optimal growth. The mathematical complications so created are the price for the greater explicitness of long run considerations thus made possible.

Sections 4-6 analyze a model with a single producible good serving both as capital in the form of a stock, and as a consumption good in the form of a flow. It is produced under a constant technology by a labor force growing exogenously at a given exponential rate. Proofs for many of the statements labeled (A), (B), ... in Section 4 are given under the same label in an Appendix.

In Section 7 the findings of the logical experiments of Sections 5, 6 are examined. The main conclusion is that some utility functions that on a priori grounds appear quite plausible and reasonable do not permit determination of an optimal growth path even in a constant technology. Tentative and intuitive explanations for this finding are offered.

Section 8 discusses in a tentative way, and without proofs, possible extensions of the analysis to a changing technology and/or a variable rate of population growth, with none, one, or both of these regarded as policy variables.

3. Pertinent Aspects of Linear and of Convex Programming

Let linear programming be applied to an allocation problem in terms of the quantities x_j , $j = 1, \ldots, n$ of a finite number n of commodities.

Then the <u>feasible set</u> D is given by a finite number of linear inequalities

(1) $$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i , \qquad i = 1, \ldots, m .$$

The objective function, or maximand, is a linear form in the x_j ,

(2) $$U = \sum_{j=1}^{n} c_j x_j .$$

The feasible set D is always closed, and may be bounded (as in Figure 1) or unbounded (Figure 2).

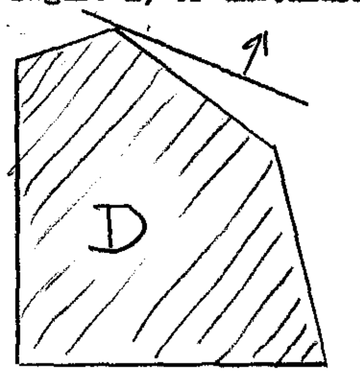

Fig. 1

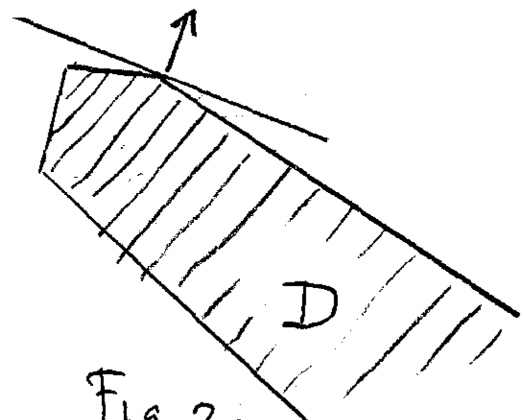

Fig. 2

The range R of the objective function on the feasible set, that is the set of values assumed by the maximand on the points of the set D is an interval. If D is bounded (contained in some hypercube), then R is necessarily also bounded. If D is unbounded, then R may still be bounded, but may also be unbounded from below, from above, or both. If R is bounded from above, an optimal point exists (Figure 2). If R is unbounded from above, no optimum exists (Figure 3). Both cases can arise on the same feasible set D through different choices of the maximand.

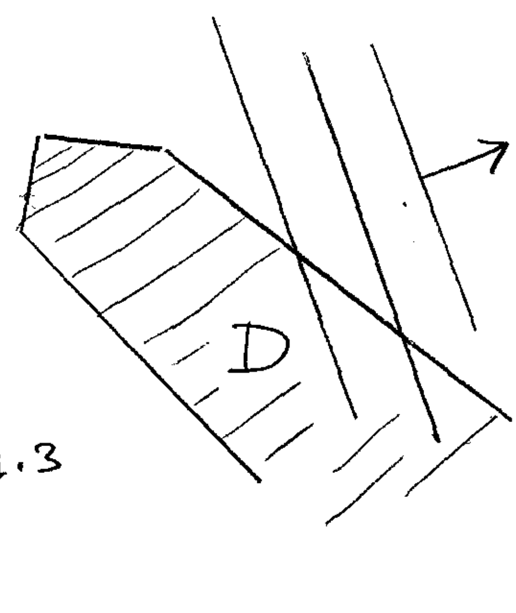

Fig. 3

A highly special form of linear programming has been used by Kantorovich [1959]. In this case the objective is defined by prescribing the ratios of the quantities of all desired goods, i.e., goods entering into the objective, and by maximizing a common scalar factor applied to these quantities (Figure 4). This problem can also be formulated in linear programming terms: One adds to the constraints (1) linear equalities expressing the prescribed ratios, and chooses as a maximand (2) the quantity of any one desired good, say.

In convex programming the feasible set is defined by

$$g_i(x_1, \dots, x_n) \geqq 0 , \qquad i = 1, \dots, n \tag{3}$$

where the g_i are concave* functions, and the maximand

$$U = U(x_1, \dots, x_n) \tag{4}$$

is another concave function (Figure 5). The term convex programming

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* A concave function $g(x_1, \dots, x_n)$ is represented by a hypersurface $y = g(x_1, \dots, x_n)$ in the space $\{y, x_1, \dots, x_n\}$ that is never "below" any of its chords (if the $+y$ direction is "up").

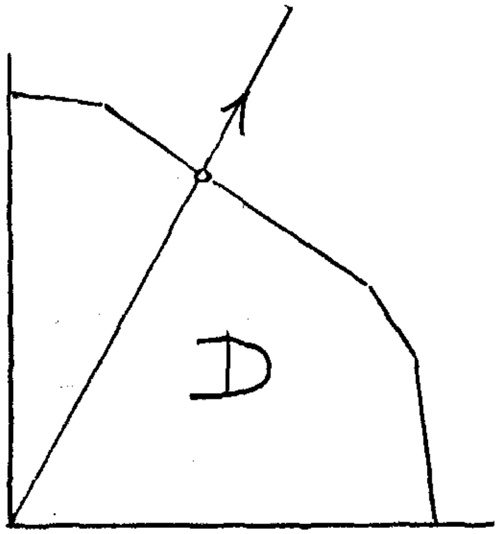

Fig. 4

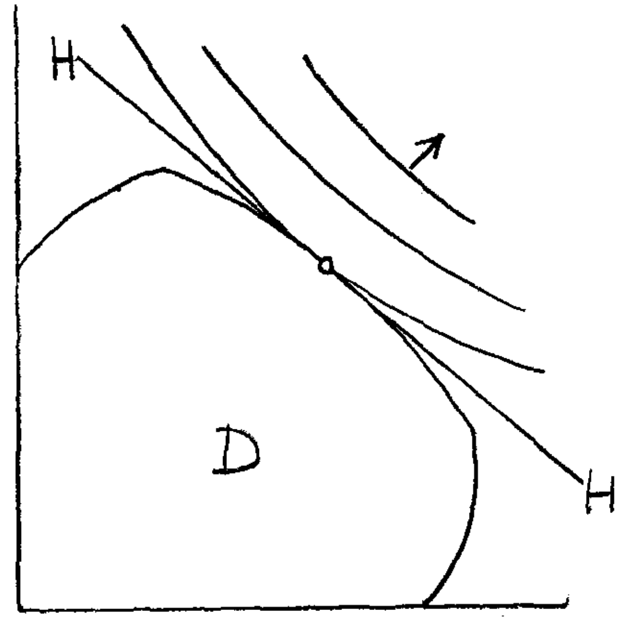

Fig. 5

derives from the fact that the feasible set, and each set of points on which the maximand attains or exceeds a given value, are convex.*

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* A convex set is a set of points containing every line segment connecting two of its points.

Linear programming is a special case of convex programming.

With any optimal point in a convex programming problem one can associate a hyperplane H through that point, which separates the feasible set from the set of points in which the maximand attains or exceeds its value in the optimal point (Figure 5). The direction coefficients of such a hyperplane define a vector of relative prices implicit in the optimal point.

One interpretation of the implicit prices is that the opening up of an opportunity to barter unlimited amounts of commodities at those relative prices does not allow the attainment of a higher value of the maximand. Moreover, if the maximand is a differentiable utility function, one may be able, by adding utility as a "commodity" and choosing its "price" to be unity, to interpret the implicit prices of the other goods as their marginal utilities either directly in consumption, or indirectly through the extra consumption made possible by the availability of one more unit of that commodity as a factor of production.

4. A One-Sector Model with Constant Technology and Steadily Increasing Labor Force

We assume that output of the single producible commodity is a twice differentiable and concave function $F(Z,L)$, homogeneous of degree one, of the capital stock Z and the size of the labor force L. These assumptions imply full employment of labor and capital, constant returns to scale, and nonincreasing returns to an increase in only one factor of production. Since capital is treated as a stock of the single producible commodity, output is at any time t to be allocated to a positive rate of consumption X_t, a positive, zero, or even negative rate of investment Y_t, and a nonnegative disposal. Hence, if we use a continuous time concept, and denote derivatives with respect to time by dots, we have

(5) $$X_t + Y_t = F(Z_t, L_t) \quad ,$$

(6) $$Y_t = \dot{Z}_t \quad .$$

$F(Z, L)$ is defined for all $Z \geqq 0$, $L \geqq 0$. We assume further that both labor and capital are essential to production, that either factor has a positive marginal productivity, and that returns to increases in only one factor are strictly decreasing,

(7)
$$\begin{cases} (7a, b) & F(0, L) = 0 \ , & F(Z, 0) = 0 \ , \\ (7c, d) & \frac{\partial F}{\partial Z} > 0 \ , & \frac{\partial F}{\partial L} > 0 \ , \\ (7e, f) & \frac{\partial^2 F}{\partial Z^2} > 0 \ , & \frac{\partial^2 F}{\partial L^2} < 0 \ . \end{cases}$$

Finally, we assume that the labor force increases at a constant positive exponential rate λ , from a given initial magnitude L_o ,

(8a, b) $$L_t = L_o \, e^{\lambda t} \quad .$$

The homogeneity of the production function enables us to go over to per-unit-of-labor-force concepts. Calling the unit of labor force briefly a "worker," let x denote consumption per worker, y ditto investment, z ditto capital stock, and

(9) $$f(z) = \frac{1}{L} F(Z, L) = F(\frac{Z}{L} \, , 1) = F(z, 1)$$

output per worker. Since we then have

$$\dot{Z}_t = \frac{d}{dt}(z_t L_t) = \dot{z}_t L_t + z_t \dot{L}_t = (\dot{z}_t + \lambda z_t)L_t ,$$

the feasible set in the space of per-worker variables x_t , z_t becomes

(10a) $$x_t + \dot{z}_t = f(z_t) - \lambda z_t ,$$

(10b,c,d) $x_t > 0$, $z_t > 0$, z_o given.

The term λz represents the net investment needed if one wants merely to supply the growing labor force with capital at the existing ratio of capital per worker.

To be specific we shall call a path (x_t, z_t) satisfying (10) attainable (for the given z_o) , and use the term feasible path in the wider sense of a path attainable for some $z_o > 0$.

It is implied in (7e) that $f(z)$ is strictly concave.*

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* A strictly concave function is one that is actually "above" all its chords.

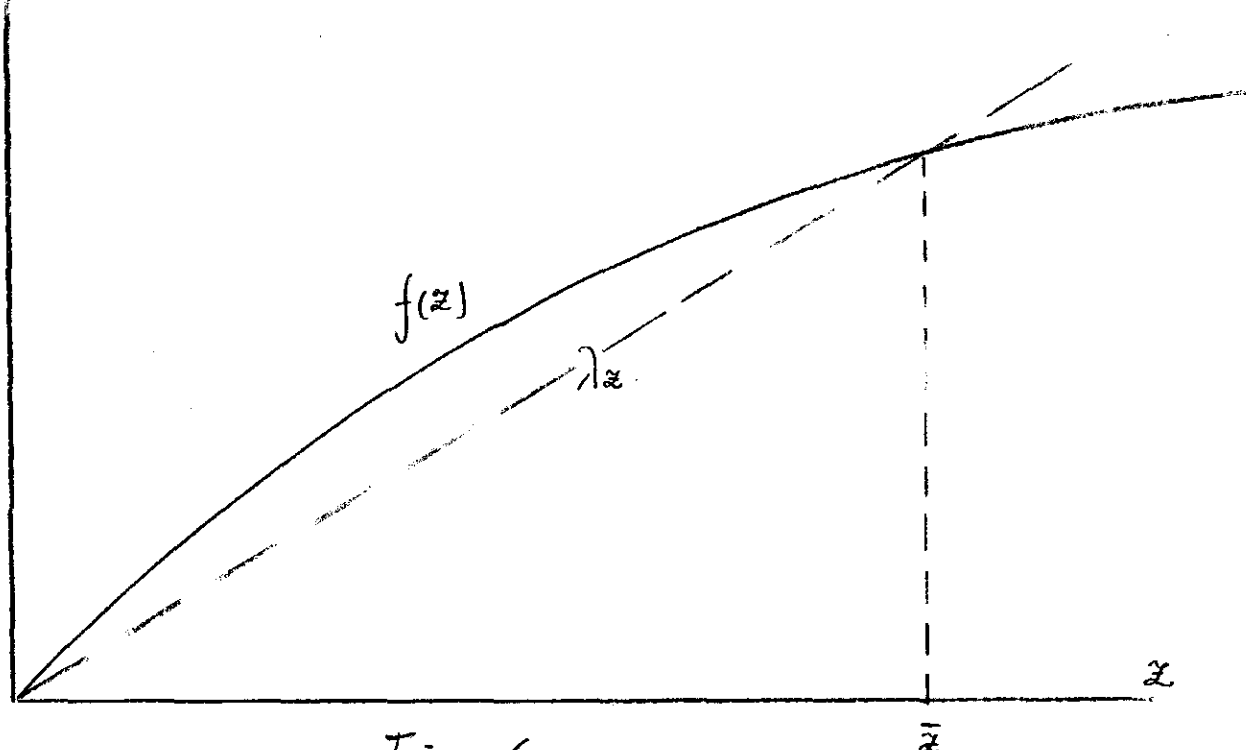

Fig. 6

The per-worker production function $f(z)$ is therefore represented by a curve such as is shown in Figure 6. The curve rises from the value $f(0) = 0$ with a decreasing slope. In particular, any line λz through the origin and of slope λ such that $0 < \lambda < f'(0)$ will ultimately intersect the curve and continue above it,*

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* To obtain (11) suppose that, for some such λ, $f(z) \geqq \lambda z$ for arbitrarily large values of z. Then, by $f(0) = 0$ and the concavity of f, $f(z) \geqq \lambda z$ for all $z \geqq 0$. But then, for any $Z > 0$, (7a) and the continuity of $F(Z, L)$ imply the contradiction $0 = F(Z, 0) =$
$= \lim_{L \to 0} F(Z, L) = \lim_{L \to 0} L\, F(\frac{Z}{L}, 1) = Z \lim_{z \to \infty} \frac{1}{z} f(z) \geqq Z\lambda > 0$.

$$
(11)\begin{cases} \text{for any } \lambda > 0 \text{ such that } 0 < \lambda < f'(0) \text{ there is a } \bar{z} > 0 \text{ such that} \\ \text{(11a)} \qquad f(z) = \lambda \bar{z} \, , \\ \text{(11b)} \qquad f(z) < \lambda z \qquad \text{for} \quad z > \bar{z} \, . \end{cases}
$$

If λ represents the rate of growth of the labor force, $\bar{z}$ represents a capital stock per worker so large that its maintencance at the same level absorbs all output, leaving nothing for consumption. If $z_o \geqq \bar{z}$, it will therefore be desirable to allow z_t to decrease at least to some level below $\bar{z}$. To avoid this uninteresting complication we shall from here on simply define "feasibility" so as to imply $0 < z_o < \bar{z}$.

Although we have not yet defined a maximand, it may be pointed out that the attainable set is now defined in a space where the "point" is a pair of functions x_t, z_t of time, defined for $0 \leqq t < \infty$. This is an infinite-dimensional space for the double reason that we use a continuous time concept and an infinite horizon. It remains infinite-dimensional if we limit ourselves* to twice differentiable functions z_t and once

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* Due to twice differentiability of the data functions $f(z)$ above and $u(x)$ below we will not be excluding any optimal paths by that requirement. However, a slightly weaker requirement will be found useful in the Appendix.

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differentiable functions x_t.

5. The Path of the Golden Rule of Accumulation

To answer an important preliminary question, we first consider a Kantorovich type restriction of the problem to a one-dimensional one. The latter problem has been formulated and solved in the last few years, independently and in one form or another, by* Allais [1962], Desroussaux [1961],

* Dates are bibliographical only and refer to the list of references below. Some of these authors used somewhat more general models involving an exponential technological improvement factor in the production function.

Phelps [1961], Joan Robinson [1962], Swan [1960], von Weizsäcker [1962].

Remove from the definition of the attainable set the restriction that z_o is given, thus making initial capital a free good. Restrict the attainable set instead by an arbitrary stipulation that consumption per worker and capital per worker are to be held constant over time,

$$x_t = x , \quad z_t = z \quad \text{for all} \quad t \geqq 0 .$$

The new "attainable" set then is given by

$$\text{(12 a,b,c)} \qquad x = f(z) - \lambda z , \qquad x > 0 , \qquad z > 0 .$$

Finally, choose z so as to maximize x, the permanent level of consumption per worker. This leads to the maximization of x

by the choice of that value $\hat{z}$ of z for which

(13 a,b) $\quad f'(\hat{z}) = \lambda , \quad$ so $\quad \hat{x} = f(\hat{z}) - \lambda\hat{z}$,

where $f'(z)$ denotes the derivative of $f(z)$.

Fig. 7

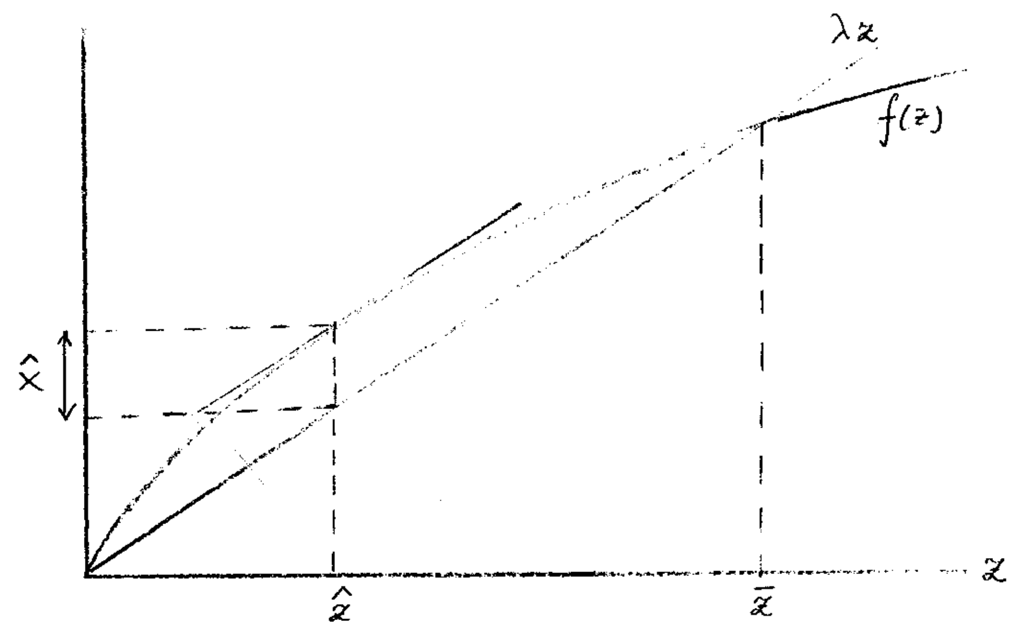

Figure 7 shows the construction. Because, of the essentiality of labor to production, i.e., assumption (7b) as reflected in (11), there is for any given slope λ such that $0 < \lambda < f'(0)$ a point $\hat{z}$ for which the tangent to the production function per worker has that slope. To interpret the condition (13a) note that, if we hold L fixed, then by the homogeneity of F ,

$$f'(z) = \frac{\partial F(Z/L,\ 1)}{\partial (Z/L)} = \frac{\partial F(Z,\ L)}{\partial Z} \quad .$$

Hence (13) expresses equality, at all times t , of the marginal productivity of capital (in producing capital, say) to the growth rate λ -- a prescription known as the golden rule of accumulation.*

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* Phelps [1961].

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6. Existence and Characteristics of Optimal Paths

We now return to the original problem that allows x_t and z_t to vary in time and recognizes the restriction (10d) of a historically given initial capital stock, and look about for a suitable maximand. We admit to an ethical preference for neutrality as between the welfare of different generations. After some hesitation, we tentatively and arbitrarily resolve another ethical conundrum by interpreting this "timing neutraility" in a per-capita sense. That is, we assume first of all that labor force and population grow in proportion. Furthermore we thus imply that, starting from the golden rule path $\hat{x}$, $\hat{z}$ of the preceding section as a base line, we welcome equally a unit increase in consumption <u>per worker</u> in any one future decade, say. Mere numbers do not give one generation an edge over another in this scheme of values.

The next difficulty we face is a technical one. A previous investigation by Koopmans [1960], continued by Koopmans, Diamond and Williamson [1962], has shown that there does not exist a utility function of all consumption paths, which at the same time exhibits timing neutraility and satisfies other reasonable postulates which all utility functions used so far have agreed with. A way out of this dilemma was shown by Ramsey [1928]. One can define an eligible set of consumption paths on which a neutral utility function can be defined. Moreover, the eligible set is a subset of the feasible set such that the remaining, ineligible, paths are clearly inferior to the eligible ones, in a sense still to be defined. In Ramsey's case, in which population was assumed stationary, the criterion of eligibility was a sufficiently rapid approach over time to what he called a state of bliss. This state was defined as either a saturation of consumers with consumption goods, or a saturation of the productive system with capital to the point where its marginal productivity has vanished -- whichever state would be encountered first. We shall find that in the present case of a steady population growth the golden rule path can take the place of Ramsey's state of bliss in defining eligibility. Thus Ramsey's device can be applied to our case with what seems a lesser strain on the imagination in regard to situations outside the range of experience.

We have one more technical choice to make. For reasons of mathematical simplicity, and at some cost in "realism," we shall model our utility function after the finite-horizon example of

$$U = \int_0^T u(x_t)\,dt \ . \tag{14}$$

As explained already in Section 1, this simple integration of an instantaneous utility flow $u(x_t)$ implies noncomplementarity between consumption in any two or more parts of the future.

We shall assume that the instantaneous utility flow is a strictly concave, increasing and twice differentiable function $u(x)$ of the instantaneous consumption flow x . This function does not change with time, and is defined for all $x > 0$. Strict concavity implies that we attribute greater weight to the marginal unit of per capita consumption of a poor generation as compared with a rich one. To assume $u(x)$ increasing rules out saturation. Finally, instead of introducing a subsistence minimum, we shall permit that

(15) $$\lim_{x \to 0} u(x) = -\infty ,$$

a strong incentive to avoid periods of very low consumption as much as is feasible. On the other hand, we do not require (15).

Let $\hat{u} = u(\hat{x})$ denote the instantaneous utility flow derived from the consumption flow per worker of the path $x_t = \hat{x}$, $z_t = \hat{z}$, of the golden rule. We shall now work with the difference between the integral (15) for any given feasible path and its value for the golden rule path, and study the behavior of this difference as T goes to infinity. The following statements can be made (for proofs see Appendix).

(A) <u>There is a number</u> $\bar{U}$ <u>such that</u>

(16) $$U_T = \int_0^T \left(u\,(x_t) - \hat{u} \right) dt \leqq \bar{U}$$

<u>for all feasible paths</u> (x_t, z_t) <u>and for all horizons</u> T .

Thus, if utility is measured in conformity with (14), no path is "infinitely better" than the golden rule path. In particular, no feasible path x_t can indefinitely maintain or exceed a utility level u in excess of $\hat{u}$. Thus the golden rule path continually attains the highest indefinitely maintainable utility flow.

(B) For every feasible path, either $\lim_{T\to\infty} U(T)$ exists (is a finite number), or $U(T)$ diverges to $-\infty$ as T tends to ∞ .

In the first case, we call the path eligible, in the second ineligible. Then (B) establishes a clear superiority of each eligible path over each ineligible one. On the eligible set we choose as the utility function

$$(17) \qquad U = \int_0^\infty \left(u(x_t) - \hat{u}\right) dt \quad .$$

It is not hard to find eligible and attainable paths for every admissible initial capital stock z_o . If $z_o > \hat{z}$, one only needs to refrain from net investment until the capital stock $\hat{Z}_t = \hat{z}\, e^{\lambda t} L_o$ of the golden rule path has caught up with the given initial stock $Z_o = z_o L_o$, and to continue along the golden rule path thereafter. If $0 < z_o < \hat{z}$, once can through a finite period of tightening the belt arrive on the same path.

(C) For any initial capital stock z_o with $0 < z_o < \bar{z}$ there exists a unique optimal path $(\hat{x}_t, \hat{z}_t)$ in the set of eligible and attainable paths. For $z_o \neq \hat{z}$, both $\hat{x}_t$ and $\hat{z}_t$ exhibit a strictly monotonic approach to $\hat{x}$ and $\hat{z}$, respectively, from below if $0 < z_o < \hat{z}$, from above if $\hat{z} < z_o < \bar{z}$. For $z_o = \hat{z}$, the optimal path is $\hat{x}_t = \hat{x}$, $\hat{z}_t = \hat{z}$ for all t , the golden rule path.

(D) The optimal path satisfies the condition

$$u'(x_t)\,\dot{z}_t = \hat{u} - u(x_t) \tag{18}$$

that at any time the net increase in capital per worker multiplied by the marginal utility of consumption per worker equals the net excess of the maximum sustainable utility level over the current utility level.
This condition is similar to the Keynes-Ramsey condition [Ramsey, 1928, equation (5)] formulated in terms of absolute amounts of consumption, and reverts to it for $\lambda = 0$. Keynes' intuitive reasoning in support of this condition carries over with only slight reinterpretation.

A number of further results can be obtained that apply equally to the case where the utility of a consumption path is defined as an integral over the instantaneous utility flow discounted at a positive instantaneous rate ρ .

(E) The utility function

$$V(\rho) = \int_0^\infty e^{-\rho t} u(x_t)\,dt\ , \quad \text{where} \quad \rho > 0\ , \tag{19}$$

is defined for all feasible paths for which $x_t \geqq \underline{x}$ for all t , whenever $\underline{x} > 0$. Ramsey's device is therefore unnecessary in this case. We shall however obtain an economy of notation if instead of $V(\rho)$ we use the utility function

$$U(\rho) = \int_0^\infty e^{-\rho t} \left(u(x_t) - \hat{u}\right) dt\ , \qquad \rho \geqq 0\ , \tag{20}$$

which differs from $V(\rho)$ by a constant if $\rho > 0$.

The stipulation in (E) that keeps consumption from becoming altogether too small is necessitated by (15), merely to prevent $U(\rho)$ from diverging to $-\infty$. However, we shall for $\rho > 0$ define as the eligible-and-attainable set the set of all paths with the prescribed z_o for which $V(\rho)$ exists. (E) assures us that no paths worth consideration are excluded from the eligible set. If z_o were to be very small, we could still allow for growth by taking $\underline{x}$ correspondingly smaller.

The following statements (F) through (J) apply equally to the cases $\rho > 0$ and $\rho = 0$. Optimality is defined by maximization of (20) on the appropriate eligible-attainable set. It is assumed in statements (F), (G) that an optimal path $(\hat{x}_t, \hat{z}_t)$ is given. The statements give economically meaningful characterizations of that path in terms of implicit prices of the consumption good and of the use of the (identical) capital good, associated with the optimal path. These prices generalize the idea of a separating hyperplane, illustrated in Figure 5, to an infinite-dimensional space. The (dated) price of the consumption good is defined from (20) by

$$(21) \qquad p_t = e^{-\rho t}\, u'(\hat{x}_t) ,$$

the present value of the marginal instantaneous utility of consumption at time t if the given optimal path is followed. The price of the use of the capital good is similarly defined by

$$(22) \qquad q_t = p_t\, g'(\hat{z}_t)$$

as the present value of the marginal productivity of capital at time t multiplied by the marginal utility of consumption at that time. Finally, we denote by

(23) $$\hat{U}(\rho) = \int_0^\infty e^{-\rho t}\left(u(\hat{x}_t) - \hat{u}\right)dt$$

the utility of the given optimal path.

(F) _If_ (x_t, z_t) _is any path, feasible or not, for which_ $U(\rho)$ _is defined, then_

(24) $$U(\rho) - \hat{U}(\rho) \leqq \int_0^\infty p_t(x_t - \hat{x}_t)dt \ .$$

Since both members vanish if $x_t = \hat{x}_t$ for all t, this means that the optimal path both

(i) maximizes utility subject to the "budget constraint"

$$\int_0^\infty p_t(x_t - \hat{x}_t)dt \leqq 0$$

(ii) minimizes "comparative consumption expenditure at implicit prices"

$$\int_0^\infty p_t(x_t - \hat{x}_t)dt$$

on the set of paths with utility equal to or exceeding that of the optimal path.

(G) _If_ (x_t, z_t) _is an eligible-attainable path_

(25) $$\int_0^\infty p_t(x_t - \hat{x}_t)dt \leqq \int_0^\infty\left(q_t(z_t - \hat{z}_t) - p_t(\dot{z}_t - \dot{\hat{z}}_t)\right)dt = \int_0^\infty (q_t + \dot{p}_t)(z_t - \hat{z}_t)dt \ .$$

Again, all three members vanish if (x_t, z_t) is itself the optimal path $(\hat{x}_t, \hat{z}_t)$. The inequality in (25), rewritten as

$$\int_0^\infty p_t(x_t + \dot{z}_t - \hat{x}_t - \dot{\hat{z}}_t)dt - \int_0^\infty q_t(z_t - \hat{z}_t)dt \leqq 0 \ ,$$

says that, at implicit prices, "revenue" from total output minus "cost of capital used" is maximized in the optimal path.

(24) and (25) together give rise to the next statement.

(H) Necessary and sufficient for the optimality of the given path $(\hat{x}_t, \hat{z}_t)$ is that the prices (21), (22) implicit in the given path satisfy the differential equation

$$(26) \qquad q_t + \dot{p}_t = 0 \qquad \text{for} \qquad t \geqq 0 .$$

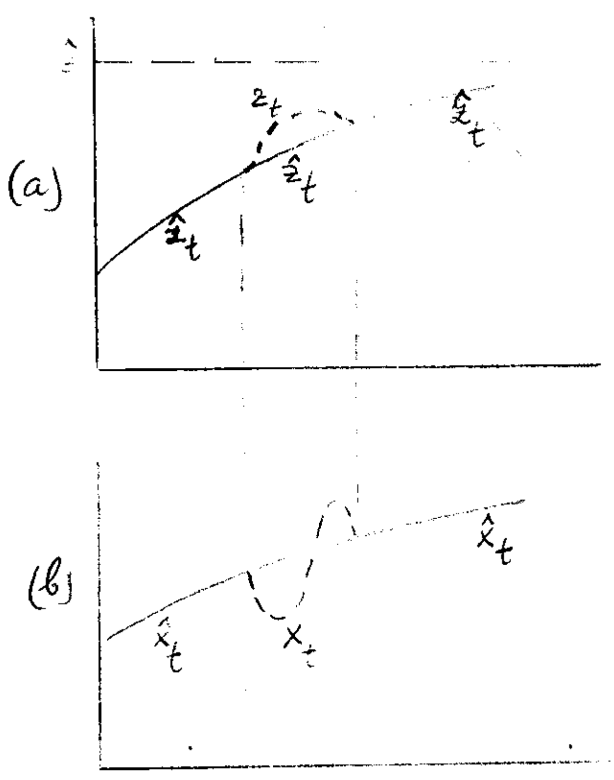


Fig. 8

To interpret this condition, let (x_t, z_t) be a path which differs from the optimal path only slightly and only on a short open interval $\mathcal{J}$, on which $z_t > \hat{z}_t$ (see Figure 8a). Then x_t will differ from $\hat{x}_t$ first because the slightly higher capital stock on $\mathcal{J}$ allows a slightly higher product, and secondly because acceleration of investment during the first part of $\mathcal{J}$ and deceleration during the second part leads to some postponement of consumption within $\mathcal{J}$. In the light of (21), (22), the condition says that, in the limit for an arbitrarily small difference $z_t - \hat{z}_t$ of arbitrarily short duration, the utility effects of these two components of $x_t - \hat{x}_t$ must cancel if the path $(\hat{x}_t, \hat{z}_t)$ is to be optimal.

Statement (H) can be used to prove the existence and uniqueness of the optimal path for $\rho \geqq 0$, and establish its main characteristics.

(I) <u>Let</u> $\hat{x}(\rho)$, $\hat{z}(\rho)$ <u>be defined as the solution</u> x, z <u>of</u>

(27) $$f'(z) = \lambda + \rho \, , \qquad f(z) - \lambda z = x \, , \quad \text{where} \quad 0 < \lambda \leqq \lambda + \rho < f'(0) \, .$$

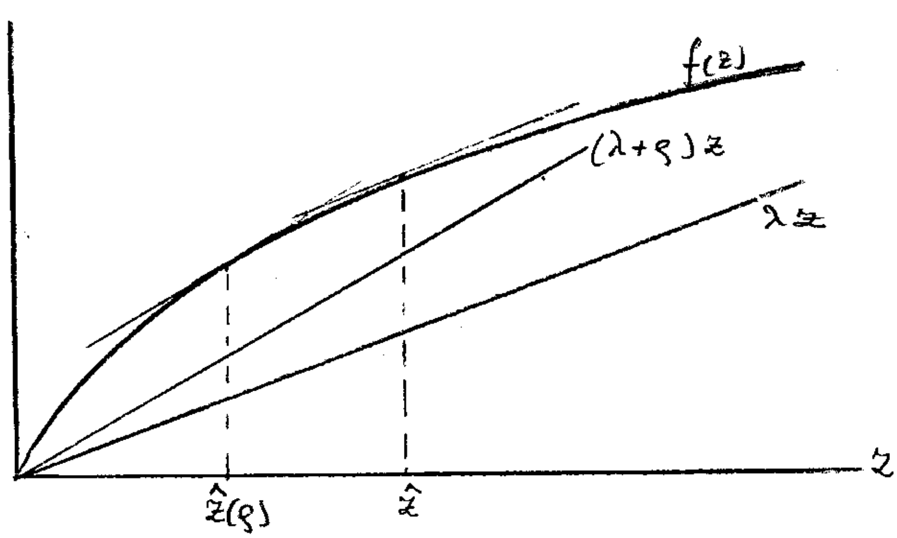
<u>Then if</u> $z_0 = \hat{z}(\rho)$, <u>the unique optimal path is</u> $\hat{x}_t = \hat{x}(\rho)$, $\hat{z}_t = \hat{z}(\rho)$ <u>for all</u> $t \geqq 0$.

The determination of $z(\rho)$ is shown in Figure 9. Because of the strict concavity of $f(z)$, $\hat{z}(\rho)$ and $\hat{x}(\rho)$ exist and are unique, and $0 < \hat{z}(\rho) < \hat{z}(\rho^*) \leqq \hat{z}$ for $\rho > \rho^* \geqq 0$, and hence, since $f(z)-\lambda z$ increases for $0 < z - \hat{z}$,

(28) $$\hat{x}(\rho) \equiv f\left(\hat{z}(\rho)\right) - \lambda\hat{z}(\rho) < \hat{x}(\rho^*) \leqq \hat{x} \qquad \text{for} \quad \rho > \rho^* \geqq 0 \, .$$

The constant optimal path made possible by the intial capital stock $z_0 = \hat{z}(\rho)$ is found to be an asymptote for the optimal paths associated with other values of z_0.

Fig. 9



(J) <u>For any initial capital stock</u> z_o <u>with</u> $0 < z_o < \bar{z}$ <u>there exists a unique optimal path</u> $(\hat{x}_t, \hat{z}_t)$. <u>For</u> $z_o \neq \hat{z}(\rho)$ <u>both</u> $\hat{x}_t$ <u>and</u> $\hat{z}_t$ <u>exhibit a monotonic and asymptotic approach to</u> $\hat{x}(\rho)$ <u>and</u> $\hat{z}(\rho)$, <u>respectively, from above if</u> $z_o > \hat{z}(\rho)$, <u>from below if</u> $z_o < \hat{z}(\rho)$.

For later discussion, we note from (28) that the asymptotic level $\hat{x}(\rho)$ of consumption per worker, while independent of the initial capital z_o, is reduced as the discount rate is increased. In particular the maximum of $\hat{x}(\rho)$ for $\rho \geqq 0$ is attained at $\rho = 0$.

Finally, a word about the case where one tries to apply a negative discount factor $\rho < 0$. Writing $-\rho = \sigma$, this means looking for a utility function extending the finite-horizon example

$$V_T(-\sigma) = \int_o^T e^{\sigma t} u(x_t)dt$$

to an infinite horizon. This is not as far-fetched as it may seem. After all, we have so far given no weight at all to mere numbers in comparing generations. If we were to weight each generation in proportion to its number, and otherwise seek neutrality with regard to timing, the population growth parameter λ would take the place of σ above.

In order to apply Ramsey's device in the present case, one would have to find a feasible path (x_t, z_t) such that

$$W_T^*(-\sigma) = \int_o^T e^{\sigma t} \left(u(x_t^*) - u(x_t)\right) dt \tag{29}$$

is uniformly bounded from above for all feasible paths and all values of T . The following statement says that no such path exists.

(K) For each attainable path (x_t, z_t), where $0 < z_o < \bar{z}$, and for each number $N > 0$, there exist another attainable path (x_t^*, z_t^*) and a number T^* such that

$$W_T^*(-\sigma) > N \quad \text{for all} \quad T \geqq T^* \tag{30}$$

This says, essentially, that there is no attainable upper bound to the range, on the attainable set, of a utility function of the type we are seeking to define. The case $\rho < 0$ is therefore analogous to the case in ordinary linear programming illustrated by Figure 4. The same difficulty was noticed and discussed by Tinbergen [1960] and by Chakravarty [1962] in connection with the case $\rho = 0$ for a model with constant returns to increases in the amount of capital alone.

In the present case, the reasons for the absence of an optimal path if $\rho < 0$ can be illustrated in terms of the path $(x_t, z_t) = (\hat{x}, \hat{z})$, optimal if $\rho = 0$ and $z_o = \hat{z}$. From (21) we see that the implicit price of the unit of consumption good per worker, associated with this path is a constant,

$$p_t = u'(\hat{x}) \quad \text{for all} \quad t .$$

This means that a sacrifice of one unit in <u>per capita</u> consumption, now made for a short period as a slight departure from this path, can be taken out by any future generation in the form of an equal augmentation of <u>per capita</u> consumption beyond that provided by the path, for a period of the same short duration. Now if either $\rho < 0$, or if $\rho = 0$ but some weight is given to population size, it will always increase utility to delay still further the time at which the fruit of the initial sacrifice is reaped.

In the proofs of the statements (A) - (K), given in the Appendix, one common characteristic of the problems considered is repeatedly used without explicit mention. At any time in an optimal path $(\hat{x}_t, \hat{z}_t)$, the capital stock $\hat{z}_t$ is the only link between the past and the future. This is due, on the one hand, to the utility function being an integral over time of instantaneous utilities (discounted or not). On the other hand, it arises from the fact that the feasibility constraint (10a) restricts $\dot{z}_t$ but not $\dot{x}_t$. Hence the function x_t is in principle free to vary discontinuously (even though it is found optimal for it not to do so). However, $\dot{z}_t$ is bounded by (10a,b), hence z_t can only vary continuously. The resulting property can be expressed formally as follows: If $(\hat{x}_t, \hat{z}_t)$ is an optimal path for given z_o, then, for any T, the path (x_t^*, z_t^*) defined by

$$x_t^* = \hat{x}_{T+t}, \qquad z_t^* = \hat{z}_{T+t}$$

is optimal for $z_o^* = \hat{z}_T$.

7. Adjusting Preferences to Opportunities

What have we learned from our "logical experiments"? We have confronted a simple model of production with a utility function representing a sum of future per-capita utilities, discounted by a positve, zero, or negative instantaneous rate of discount ρ . We have found that $\rho = 0$ is the smallest rate for which an optimal path exists.

Let us assume for the sake of argument that the present model is representative enough to be looked on as a tentative test of the applicability of the ethical principles under consideration. Then we have just managed to avoid discriminating against future generations on the basis of remoteness of the time at which they live. However, this close escape for virtue was possible only by making welfare comparisons on a per capita basis. If instead we should want to weight per capita welfare by population size, then we are forced to discriminate on the basis of historical time by positive discounting. There seems to be no way, in an indefinitely growing population, to give equal weight to all individuals living at all times in the future.

This dilemma suggests that the open-endedness of the future imposed mathematical limits on the autonomy of ethical thought. The suggestion may come as a shock to welfare economists, because no such logical obstacles have been encountered in the more fully explored problems of allocation and distribution for a finite population. It is true that the mere fact that we are considering an infinite number of people does not fully explain the dilemma. For Ramsey was able, albeit by artificial assumptions, to indicate a fair solution to the problem for the infinite future of a population of constant size. Our difficulty is therefore connected with the assumption of an indefinite growth in the population.

The following reasoning may further illuminate the reasons for the nonexistence of an optimal path with negative ρ . Assume that $0 > \rho > -\lambda$. (Of course, $\rho = -\lambda$ would correspond to equal weights given to the utilities of all individuals. Our illustration is clearer if we do not go quite that far). Consider now an optimal path for the finite time period $0 \leqq t \leqq T$, defined by initial and terminal per worker capital stock levels $z_0 = z_T = \hat{z}$ both equal to that level $\hat{z}$ which, <u>if maintained at all times</u>, would secure the maximum maintainable consumption per head. The analysis associated with the proofs of (H), (I), (J) in the appendix now indicates that, if the level $\hat{z}$ is prescribed only for $t = 0$ and $t = T$, the optimal

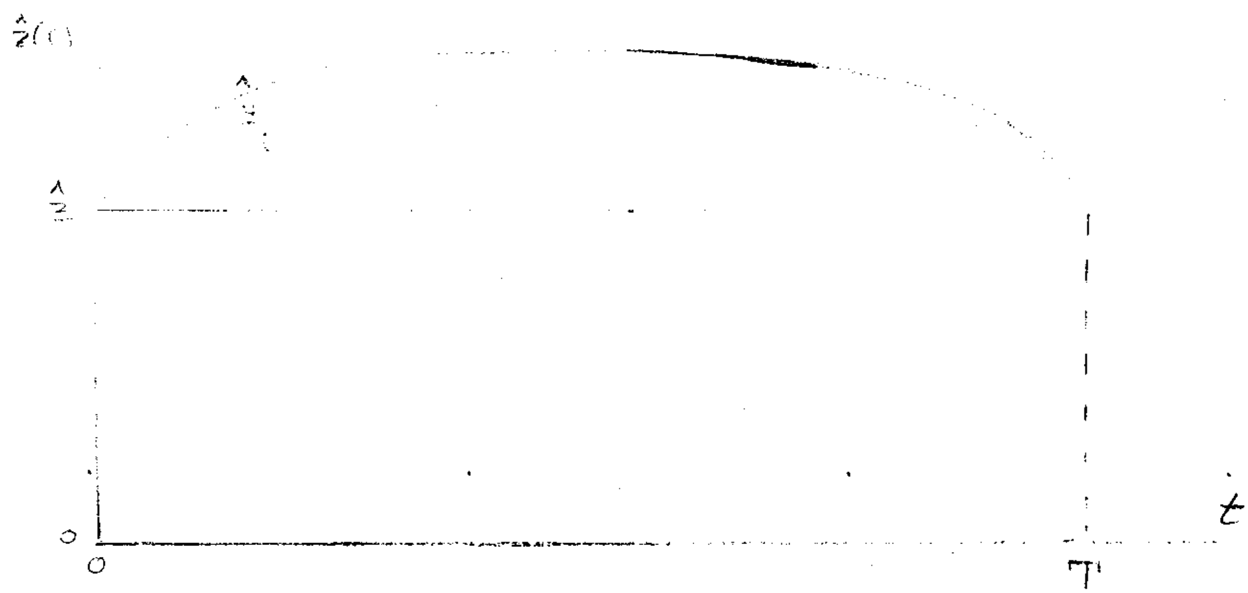


Fig. 10

path bulges out toward the level of $\hat{z}(\rho)$, as indicated in Figure 10. The interpretation is roughly as follows. The negative discount rate gives the greatest weight to the per capita utility of the last generation living within the planning period $[0, T]$. In response to this weighting system, the optimal path provides for a reduction in capital per worker (a "disinvestment" in the per capita sense) during a terminal segment of the planning period, in order to allow for high consumption at that time. To make this possible, all preceding generations make a sacrifice. For the first generation, this takes the form of heavy investment needed to increase the capital stock more than in proportion to population growth. For the intermediate generations, it consists in approximately maintaining the capital stock -- by continued proportional growth -- at a per capita level in excess of that which would maximize per capita consumption.

Now if T is increased, the benefited generation becomes a more and more distant one. If $T = \infty$, there is no benefited generation, and the limiting position of the curve in Figure 10, while mathematically well-defined, merely describes a path of indefinite and fruitless sacrifice.

The problem appears in even sharper light if technological progress is also recognized. A study by Inagaki [1963] uses a Cobb-Douglas production function

$$F(Z, L, t) = \text{const. } e^{\beta t} Z^{\alpha} L^{1-\alpha}$$

subject to exogenous technological progress at the constant proportional rate β , an instantaneous utility function

$$u(x) = \int^{x} \frac{ds}{\log s - \log \underline{x}} \tag{31}$$

exhibiting suitable behavior for large values of x , and a labor force growing exponentially at the rate λ . Among other results, Inagaki finds that, for the integral $V(\rho)$ as defined in (19) to converge on the counterpart of our path $(x_t, z_t) = \left(\hat{x}(\rho), \hat{z}(\rho)\right)$, it is necessary that

$$\rho > \frac{\beta}{1-\alpha} . \tag{32}$$

Let us assume that Ramsey's device can be used also in this case, and that it would again merely result in adding the borderline value $\rho = \frac{\beta}{1-\alpha}$ to the set of discount rates defining a utility function for which an optimal path exists. Then a predictable positive lower bound to the rate of technical progress, valid for an indefinite future period, precludes application of the ethical principle of timing neutrality in per capita utility -- not to speak at all of weighting generations by their numbers.

Thus, if in the face of technological progress we want to hold on to the idea of maximizing a utility integral such as (31) over time, we must invent a discount rate ρ satisfying (32), or its equivalent for another production function. Such a discount rate might just have to be a pragmatic one having no basis in a priori ethical thought. While it might well be a result, conscious or unconscious, of political processes or decisions, it would have to be revised upward if it is estimated that technological progress will accelerate to such an extent as to "overtake it," and could be revised downward if it is expected that progress will slow down.

One might instead conclude that the whole idea of maximizing a utility integral is not flexible enough to fit the inequality of opportunity between generations inherent in modern technology. Two alternative notions have been partially explored by the present author, using a discrete concept of time. In one of these [Koopmans, 1960], the utility function of a consumption path x_t , $t = 1, 2, \ldots,$ can be defined by a recursive relation

$$U(x_1, x_2, \ldots) = V\left(u(x_1), U(x_2, x_3, \ldots)\right)$$

in terms of a one-period utility function $u(x)$ and an aggregator function $V(u, U)$. This formulation allows the (scale-invariant) discount factor

$$\left(\frac{\partial V(u, U)}{\partial U}\right)_{u=u(x),\ U = U(x, x, \ldots)}$$

associated with a constant path to increase or decrease with the level x at which the path proceeds. The second alternative [Koopmans, 1962] is an attempt to express formally the idea of a present preference for flexibility in future preferences between different commodity bundles of the same timing, or between physically the same bundles spread out differently over time, or between bundles differing in both respects. Further analysis will be needed to determine whether the first idea is sufficiently flexible to enable us to avoid the difficulties we have encountered, or, if not, whether the second idea can be made workable.

8. Technical Progress and Population Growth as Possible Policy Variables

So far we have treated both technical progress and population growth as exogenously given. It should now be recognized that both variables can be, and are in many countries, influenced by public and private policies and attitudes. Technical change is furthered by government conduct or support of research and of education, by the tax treatment of depreciation and obsolescence, and by business policies with regard to research and development. Population growth is influenced by expenditures for public health, by family allowances, by government policies toward family planning, and by general cultural and religious attitudes toward the idea of population control. In addition, both variables are in part endogenously affected by the level of income.

Both possibilities of partial control raise new conceptual problems in formalizing the idea of optimal economic growth. In the middle of the scientific explosion, it is hard to assess whether technological progress can go on forever, so that also its rate can be raised or lowered forever. Alternatively, a higher rate of discovery and invention now might entail a lower rate of progress at some later time when the fund of knowledge usable in production nears completion. Another consideration is that technological progress raises transition and dislocation difficulties that affect the relative welfare of different individuals within the same generation.

The possibility of influencing population size raises the question of the value of population size in itself -- as distinct from the question of the weight given to numbers in aggregating utility over generations, discussed above. It should be noted that all utility functions discussed in this paper imply neutrality with regard to population size as such. The question is of some importance because a different attitude might lead to a different balance between the "value of numbers" and the loss of per capita income that may result from the ratio of population to land and/or other resources. This problem did not come up in the more formal analysis of the preceding section because the assumption of constant returns to proportional increases in both labor and capital precluded the recognition of resource limitations.

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